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Stokes's first problem for a dipolar fluid with nonclassical heat conduction

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Abstract. The classical heat law of Fourier associates an infinite speed of propagation to a thermal disturbance in a material body. Such behavior is a violation of the causality principle. In recent years, several modifications of Fourier's heat law have been proposed. In this work a modification of Fourier's heat law based on the Maxwell-Cattaneo-Fox (MCF) model is used to describe the influence of heat conduction at low temperatures and/or high heat-flux conditions on Stokes' first problem for a dipolar fluid. The effects of discontinuous boundary data and a finite propagation speed of thermal waves on the velocity and stress fields are investigated. In addition, special and limiting cases of the material constants are examined. Lastly, results for the special case of equal dipolar constants are compared to the corresponding results found using Fourier's heat law.

Key words: dipolar fluids, discontinuities, Laplace transform, Maxwell-Cattaneo-Fox model.

1. Introduction

According to Fourier's heat law, thermal conduction in a homogeneous and isotropic medium is governed by the phenomenological equation

$$\mathbf{q} = -\kappa \nabla \theta, \tag{1.1}$$

where $\mathbf{q}$ is the heat flux vector, $\kappa > 0$ is the constant thermal conductivity of the medium, and θ is the absolute temperature distribution. When combined with the conservation of energy law, and when the pressure and density gradients are assumed to be zero, Fourier's heat law results in the parabolic heat-conduction equation

$$\frac{\partial \theta}{\partial t} = \frac{\kappa}{\rho c_p} \nabla^2 \theta, \tag{1.2}$$

where ρ is the density and c_p is the specific heat at constant pressure. Results obtained from Equation (1.2) are generally in close agreement with experimental data for temperatures well above absolute zero (room temperature for example). However, the parabolic nature of this equation implies an infinite speed of heat propagation, thus violating the principle of causality.

Over the years, several researchers have proposed modifications to Fourier's heat law in an effort to overcome the propagation speed defect. In 1867, Maxwell [1] derived the first generalization of Fourier's heat law. The first term of his equation [1, Equation (143)] corresponds to the time derivative of the heat flux vector multiplied by a constant relaxation time (which is termed τ_0 in this article). In Maxwell's work τ_0 was of a very small magnitude. He therefore took it to be zero. In justification he remarked, 'The first term of this equation (143) may be neglected, as the rate of conduction will rapidly establish itself'. Had Maxwell considered a nonzero relaxation time, his modification of Fourier's heat law would have been the first to give a finite speed of heat propagation. In 1944, Peshkov [2] was the first to observe thermal waves (or second sound) propagating in liquid helium II. From his observations he concluded that in liquid helium II at 1·4° K the average velocity of the second sound is 19m/sec. In 1948, Cattaneo [3] was the first to offer an explicit mathematical correction of the propagation speed defect inherent in Fourier's heat conduction law. Cattaneo's theory allows for the existence of thermal waves which propagate at finite speeds. In Cattaneo's theory these waves are the means by which heat flow occurs in gases. Cattaneo's heat law (or Cattaneo's equation) is expressed as

$$\tau_0 \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa \nabla \theta. \tag{1.3}$$

For $\tau_0 = 0$, Equation (1.3) reduces to Fourier's heat law. Cattaneo's heat conduction law results in the hyperbolic equation

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{1}{\tau_0} \frac{\partial \theta}{\partial t} = \frac{\kappa}{\tau_0 C_v} \nabla^2 \theta, \tag{1.4}$$

where C_v is the heat capacity per unit volume and $\tau_0 > 0$. Equation (1.4) is a special case of the general telegraph equation and is known as the dissipative or damped wave equation (see Jou *et al.* [4, pp. 167– 200]). Chester [5], in 1963, stated that wave propagation will dominate when

$$\left| \frac{\partial \theta}{\partial t} \right| \gg \left| \frac{\theta}{\tau_0} \right|, \tag{1.5}$$

and diffusion will dominate when the inequality is reversed. Over twenty years after Peshkov first detected the second sound effect, Ackerman *et al.* [6] where the first to successfully measure the speed of the temperature waves in solid helium. In 1984, Straughan and Franchi [7] investigated the question of convective stability in the Bénard problem when the Maxwell-Cattaneo (or MCF) heat law is used. The same problem was also analyzed by McTaggart and Lindsay [8] in 1985. They demonstrated that there exists a significant difference in results when the MCF model is used in place of Fourier's heat law. In the MCF model, the evolution of the heat flux vector is described by the equation

$$\tau_0 (\dot{q}_i - \omega_{ij} q_j) = -q_i - \kappa \theta_{,i}, \tag{1.6}$$

where ω_{ij} is the vorticity. For $\omega_{ij} = 0$ and $\dot{q}_i = \partial \mathbf{q}/\partial t$, the MCF model reduces to the Cattaneo equation. In 1995, Puri and Kythe [9] investigated the effects of using the MCF model in Stokes's second problem for a viscous fluid. In 1997, Puri and Kythe [10] also studied the effects of the MCF model and discontinuous boundary data on the velocity gradients and temperature fields occurring in Stokes' first problem for a viscous fluid. They note that in the theory of generalized thermoelasticity, the nondimensional thermal relaxation time λ (defined as $\lambda = CP$, where C and P are the Cattaneo and Prandtl numbers respectively) is of the order 10^{-2} (see also Puri [11] where it is defined as m). A detailed history of heat conduction theory is given by Joseph and Preziosi in [12] and [13]. In addition to discussing various other models of heat conduction, these authors state that Cattaneo's equation is the most obvious and simple generalization of Fourier's law that gives rise to finite speeds of propagation. In their review article, Dreyer and Struchtrup [14] discuss low temperature heat propagation in dielectric solids where second effects are present. In addition, they point out that at low temperatures Fourier's heat law 'becomes measurably false'. Chandrasekharaiah [15] notes that Cattaneo's heat law should be used in both very low temperature ($\approx 1^\circ$ K) and high heat flux ($> 10^9$ W/m^2) applications.

It is generally known that in the nonlinear framework, Cattaneo's equation does not satisfy the entropy principle of thermodynamics. To resolve this, Coleman *et al.* [16] have shown that it is necessary that the internal energy depends not only on θ, but also on $\mathbf{q}$. This approach, however, has led to some questionable results [13, 17]. A more realistic model for heat conductors comes from Morro and Ruggeri [17]. They have proposed that heat conduction in solids is governed by the equation

$$\tau(\theta)\frac{\partial \mathbf{q}}{\partial t} + \left(1 + \Upsilon(\theta)\frac{\partial \theta}{\partial t}\right)\mathbf{q} = -\kappa \nabla \theta, \tag{1.7}$$

where $\tau(\theta)$ is a temperature-dependant relaxation time. Equation (1.7) is a nonlinear generalization of Cattaneo's equation which fits the experimental data and is compatible with the requirements of thermodynamics (see also Ruggeri *et al.* [18]).

In this work we consider the influence of the MCF model on a dipolar fluid, a common example being liquid sulphur dioxide [19]. Dipolar fluids can be considered as special cases of fluids with deformable microstructure (Cowin [20]). According to Erdogan [21], this microstructure may consists of such entities as bubbles, atoms, particulate matter, ions or other suspended bodies. In 1967, Bleustein and Green [22] presented the theory of dipolar fluids, the simplest examples of a class of non-Newtonian fluids known as multipolar fluids. Green and Naghdi [23] proposed an alternative form of dipolar inertia to that given in [22]. Ariman *et al.* [24] note that in dipolar fluid theory, the second order gradient of the velocity vector is inserted into the stress constitutive equations. Thus, dipolar fluid theory gives one vector equation to describe the velocity field. As a result not all components of the stress and couple stress tensors are known. Guram [25] has solved Stokes's first problem (see Schlichting [26, pp. 72–73]) for a dipolar fluid for the special case of dipolar constants $d = l$. Saran [27] investigated both Couette and Poiseuille flows of a dipolar fluid through a porous channel. Straughan [28] studied the nonlinear stability problem in the case where a layer of dipolar fluid is heated from below. Jordan [29] studied Stokes's first and second problem for a dipolar fluid under the MCF model for the case of equal dipolar constants. Puri and Jordan [30] have investigated Stokes's second problem for a dipolar fluid, also using the MCF model, for arbitrary values of the dipolar constants.

One can consider this article as both a generalization of the research of Puri and Kythe [10] to dipolar fluids and as an extension and refinement of the work of Jordan [29]. Our motivation in doing this work stems from the ever growing number of low-temperature and/or high heat flux applications of non-Newtonian fluids in areas such as medical research, space exploration, and low-temperature physics. Lastly, we must note that in the general case of thermoviscous fluids, particularly monoatomic gases, a complicated mutual interaction between temperature and velocity fields exist (see Müeller and Ruggeri [31, pp. 1–61]). Thus, because of the linear nature of the problem presented here, this work should be considered as only a first approximation to a more complex problem.

2. Mathematical formulation

Taking the z-axis of a cartesian coordinate system in the upward direction, let an incompressible dipolar fluid fill the space $x > 0$ adjacent to a flat vertical plate occupying the yz-plane. Initially, both fluid and plate are at rest and at constant temperature θ_∞, the fluid's free stream temperature. The flow is induced by heating of the plate in the form $(\theta_w - \theta_\infty) f(t)$, where θ_w is some constant; or by the motion of the plate along the z-axis with velocity $U_0 g(t)$, where U_0 is a constant; or both. Furthermore, both $f(t)$ and $g(t)$ are zero for time $t < 0$. Under these conditions no flow occurs in the x and y-directions and the flow velocity at a given point in the fluid depends only on its distance from the plate and the time.

The basic equations of continuity, momentum, and energy governing an isotropic, homogeneous, incompressible dipolar fluid as given by Bleustein and Green [22] and employing the form of dipolar inertia proposed by Green and Naghdi [23] are, under the Boussinesq approximation, given by

$$v_{i,i} = 0, \tag{2.1}$$

$$\begin{aligned} &\mu(1 - l^2\nabla^2)\nabla^2 v_k + \rho(F_k - \mathcal{F}_{jk,j}) - p_{,k} - \rho[1 - \beta(\theta - \theta_\infty)]\mathrm{g}\delta_{k3} \\ &\quad = \rho(1 - d^2\nabla^2)\dot{v}_k + \rho d^2(v_{k,i}v_{i,j} + v_{k,i}v_{j,i} - v_{i,k}v_{i,j})_{,j}, \end{aligned} \tag{2.2}$$

$$\rho(\dot{A} + \dot{\theta}S + \theta\dot{S}) - \rho r = -q_{i,i} + \tau_{ik}d_{ik} + \Sigma_{(ki)j}A_{jik}, \tag{2.3}$$

where the vector $\mathbf{v} = (0, 0, u(x, t))$ represents the velocity, $\theta = \theta(x, t)$, μ the dynamic viscosity, p the pressure, g the gravitational acceleration, β the coefficient of thermal expansion, τ_{ik} the stress tensor, d_{ik} the strain tensor, d and l are nonnegative material constants with the dimensions of length (termed the dipolar constants), F_k and $\mathcal{F}_{jk}$ are, respectively, the monopolar (macroscopic) and dipolar (microscopic) body forces per unit mass, r is the heat supply function per unit mass per unit time, $A = A(\theta)$ is the Helmholtz free energy function, $S = -\partial A/\partial\theta$ is the entropy, and δ_{ij} is the Kronecker delta. Furthermore, commas denote partial differentiation with respect to the space coordinates, dots represent material derivatives, and the summation convention has been employed. The constitutive equations for the stress tensor τ_{ik} [22], the dipolar stress tensor Σ_{ijk} [22], and the heat flux vector q_i of the MCF theory for dipolar fluids are

$$\tau_{ij} + \Phi\delta_{ij} = 2\mu d_{ij}, \tag{2.4}$$

$$\Sigma_{(ij)k} + \Psi_i\delta_{jk} + \Psi_j\delta_{ik} = h_1\delta_{ij}A_{kmm} + h_2(A_{ijk} + A_{jik}) + h_3A_{kji} + \gamma\delta_{ij}\theta_{,k}, \tag{2.5}$$

$$\tau_0(\dot{q}_i - \omega_{ij}q_j) = -q_i - \kappa\theta_{,i} + \alpha A_{ikk}, \tag{2.6}$$

where

$$\tau_{ij} \equiv \sigma_{ij} + \Sigma_{kij,k} + \rho(\mathcal{F}_{ij} - \Gamma_{ij}) = \tau_{ji}, \tag{2.7}$$

$$A_{ijk} = v_{i,jk} = A_{ikj}, \tag{2.8}$$

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) = d_{ji}, \tag{2.9}$$

the arbitrary functions Φ and Ψ_i govern the pressure and arise from the solenoidal nature of the velocity field, $h_\ell(\ell = 1, 2, 3)$ are material constants, α and γ are also material constants which provide thermomechanical coupling [22], $\Sigma_{(ij)k}$ are the components of the dipolar stress tensor which are symmetric in the first two indices, σ_{ij} is the monopolar stress tensor, and Γ_{ij} is the dipolar inertia. The dipolar inertia given by Bleustein and Green [22] and the alternative form of dipolar inertia proposed by Green and Naghdi [23] are

$$\Gamma_{ij} = d^2(\dot{v}_{j,i} - v_{j,k}v_{k,i}), \tag{2.10}$$

$$\Gamma_{ij} = d^2(\dot{v}_{j,i} - v_{j,k}v_{k,i} - v_{j,k}v_{i,k} + v_{k,i}v_{k,j}), \tag{2.11}$$

respectively. The pressure p and dipolar constant l^2 are defined as

$$p \equiv \Phi - 2\Psi_{i,i}, \qquad l^2 \equiv \frac{h_1 + h_3}{\mu} \geqslant 0. \tag{2.12}$$

Lastly, the material constants satisfy the following inequalities:

$$\begin{aligned} &\mu \geqslant 0, \qquad h_1 + h_3 \geqslant 0, \qquad 2h_2 + h_3 \geqslant 0, \qquad h_3 - h_2 \geqslant 0, \\ &\quad 5h_1 - h_2 + 2h_3 \geqslant 0. \end{aligned} \tag{2.13}$$

Based on the arguments of Guram [25], the equation of motion reduces to

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} - d^2\frac{\partial^3 u}{\partial x^2 \partial t} + \nu l^2 \frac{\partial^4 u}{\partial x^4} = \mathrm{g}\beta(\theta - \theta_\infty), \tag{2.14}$$

where $\nu = \mu/\rho$ is the kinematic viscosity and body forces have been neglected. Since $A_{ikk} = 0$ in this problem, the heat conduction equation is given by [10]

$$\tau_0\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} = \frac{\kappa}{\rho c_p}\frac{\partial^2\theta}{\partial x^2}. \tag{2.15}$$

We now introduce the following nondimensional quantities:

$$\begin{aligned} &x' = \frac{U_0}{\nu}x, \qquad u' = \frac{u}{U_0}, \qquad t' = \frac{U_0^2}{\nu}t, \qquad \theta' = \frac{\theta - \theta_\infty}{\theta_w - \theta_\infty}, \qquad G = \frac{\nu\,\mathrm{g}\beta(\theta_w - \theta_\infty)}{U_0^3}, \\ &P = \frac{\nu\rho c_p}{\kappa}, \qquad C = \frac{\kappa\tau_0 U_0^2}{\nu\rho c_p}, \qquad \lambda = \frac{\tau_0 U_0^2}{\nu} = CP, \\ &c = \sqrt{\lambda P}, \qquad l_1 = \frac{dU_0}{\nu}, \qquad l_2 = \frac{lU_0}{\nu}, \end{aligned} \tag{2.16}$$

where G is a modified Grashof number and c is defined here for future convenience. Using (2.16), we may write the equations of motion and heat conduction as follows:

$$u_t - u_{xx} - l_1^2 u_{xxt} + l_2^2 u_{xxxx} = G\theta, \tag{2.17}$$

and

$$\lambda P\theta_{tt} + P\theta_t = \theta_{xx}, \tag{2.18}$$

where variable subscripts on u and θ denote partial differentiation and the primes have been suppressed. The nondimensional boundary conditions are

$$\theta(0,t) = f(t), u(0,t) = g(t), \theta(\infty,t) = u(\infty,t) = 0,$$

$$u_{xx}(0,t) = M_1, u_{xx}(\infty,t) = 0, \tag{2.19}$$

where M_1, a constant, is the nondimensional dipolar stress at the plate. The nondimensional initial conditions are

$$\theta(x,0) = \theta_t(x,0) = u(x,0) = 0. \tag{2.20}$$

Applying the Laplace transform with respect to time to Equations (2.17)–(2.19), and invoking (2.20), we obtain

$$l_2^2 \bar{u}_{xxxx} - (1 + sl_1^2)\bar{u}_{xx} + s\bar{u} = G\bar{\theta}, \tag{2.21}$$

and

$$\lambda P s^2 \bar{\theta} + Ps\bar{\theta} = \bar{\theta}_{xx}, \tag{2.22}$$

where s is the transform parameter and a bar over a quantity denotes the corresponding quantity in the transform domain. The temperature field in the transform domain is given in [10] as

$$\bar{\theta}(x,s) = \bar{f}(s)\ \exp[-x\sqrt{\lambda P s^2 + Ps}]. \tag{2.23}$$

Thus, the velocity field in the transform domain is

$$\bar{u}(x,s) = \bar{u}_1(x,s) + \bar{u}_2(x,s) + \bar{u}_3(x,s), \tag{2.24}$$

where

$$\bar{u}_1 = \frac{l_2^2}{l_1^2\sqrt{(s-\chi)^2+\xi}}\left[\frac{M_1}{s}\{\mathrm{e}^{-r_2x} - \mathrm{e}^{-r_1x}\} - \bar{g}(s)\{\mathrm{e}^{-r_2x}r_1^2 - \mathrm{e}^{-r_1x}r_2^2\}\right], \tag{2.25}$$

$$\bar{u}_2 = \frac{-G\bar{f}(s)}{s\lambda^2P^2l_1^2\Pi(s)\sqrt{(s-\chi)^2+\xi}}$$

$$\times\left[(\lambda P s^2 + Ps)\{\mathrm{e}^{-r_2x} - \mathrm{e}^{-r_1x}\} - \{\mathrm{e}^{-r_2x}r_1^2 - \mathrm{e}^{-r_1x}r_2^2\}\right], \tag{2.26}$$

$$\bar{u}_3 = \frac{G\bar{f}(s)}{s\lambda^2P^2l_2^2\Pi(s)}\ \exp[-x\sqrt{\lambda P s^2 + Ps}], \tag{2.27}$$

and where

$$\Pi(s) = s^3 + s^2\left(\frac{2Pl_2^2 - l_1^2}{\lambda P l_2^2}\right) + s\left(\frac{Pl_2^2 - l_2^2 - \lambda}{\lambda^2 P l_2^2}\right) + \frac{(1-P)}{\lambda^2 P^2 l_2^2}, \tag{2.28}$$

$$\chi = \frac{2l_2^2 - l_1^2}{l_1^4}, \qquad \xi = \frac{4l_2^2(l_1^2 - l_2^2)}{l_1^8}, \qquad r_{1,2} = \frac{l_1}{l_2}\sqrt{\frac{s + l_1^{-2} \mp \sqrt{(s-\chi)^2 + \xi}}{2}}, \tag{2.29}$$

and we now impose the additional conditions $l_1, l_2, \lambda > 0$.

3. Time-domain solutions

The temperature field solution $\theta(x, t)$ will not be given here since it already appears in [10]. In this section we invert Equations (2.25)–(2.27) for both impulsive and Heaviside-type boundary data. Hence, taking $f(t) = g(t) = \delta(t)$ (*i.e.*, $\bar{f}(s) = \bar{g}(s) = 1$), where $\delta(\cdot)$ is the Dirac delta function, and using the inverse Laplace transform theorem, in conjunction with the tables of inverse Laplace transforms given in [32, pp. 227–250], we find

$$\begin{aligned} u_1 \;=\; \delta(t)\Bigg\{ & \tfrac{1}{2}(\mathrm{e}^{-x/l_2} + 1) - \frac{1}{2\pi}\int_0^\infty \frac{\sin[x\mathcal{P}(\eta)]}{\eta}\,\mathrm{d}\eta + \frac{1}{2\pi}\int_0^\infty \frac{\sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta \\ & + \frac{M_1 l_2^2}{\pi l_1^2}\int_0^\infty \frac{\mathrm{e}^{-\eta t}\,\sin[x\mathcal{P}(\eta)]}{\eta\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta + M_1 l_2^2(\mathrm{e}^{-x/l_2} - 1)\Bigg\} \\ + H(t)\Bigg\{ & \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-\eta t}\,\sin[x\mathcal{P}(\eta)]\mathrm{d}\eta \\ & - \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{e}^{-\eta t}(\eta - l_1^{-2})\sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\mathrm{d}\eta\Bigg\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} u_2 \;=\; H(t)\Bigg\{ & \frac{G}{\lambda^2 P l_1^2 \pi}\Bigg[-\lambda\int_0^\infty Q_3(\eta,t)\frac{\eta\,\mathrm{e}^{-\eta t}\,\sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta \\ & + \lambda\Lambda_3(t)\int_0^\infty \frac{\sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta \\ & + \int_0^\infty Q_3(\eta,t)\frac{\mathrm{e}^{-\eta t}\,\sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\Bigg] \\ & - \frac{G}{\lambda^2 P^2 l_2^2}\Bigg[\Lambda_4(t) - \frac{1}{2\pi}\int_0^\infty Q_3(\eta,t)\frac{\mathrm{e}^{-\eta t}\,\sin[x\mathcal{P}(\eta)]}{\eta}\,\mathrm{d}\eta \end{aligned}$$

$$+\frac{1}{2\pi}\int_0^\infty Q_3(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta$$

$$\left.\left.-\frac{1}{2\pi l_1^2}\int_0^\infty Q_3(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\eta\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\right]\right\}, \tag{3.2}$$

$$u_3 = H(t-cx)\left\{\frac{G}{\lambda^2P^2l_2^2}\left[\Lambda_4(t-cx)\exp\left(\frac{-cx}{2\lambda}\right)+cx\int_{cx}^t\Lambda_4(t-\zeta)B(x,\zeta)\mathrm{d}\zeta\right]\right\}, \tag{3.3}$$

and taking $f(t)=g(t)=H(t)$ (*i.e.*, $\bar{f}(s)=\bar{g}(s)=1/s$), where $H(\cdot)$ is the Heaviside unit step function, we have

$$u_1 = H(t)\left\{1-\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\eta}\,\mathrm{d}\eta+\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\right.$$

$$\left.+\frac{(2M_1l_2^2-1)}{2\pi l_1^2}\int_0^\infty\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\eta\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta+M_1l_2^2(\mathrm{e}^{-x/l_2}-1)\right\}, \tag{3.4}$$

$$u_2 = H(t)\left\{\frac{G}{\lambda^2Pl_1^2\pi}\left[\lambda\int_0^\infty Q_3(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\right.\right.$$

$$\left.+\int_0^\infty Q_4(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\right]$$

$$-\frac{G}{\lambda^2P^2l_2^2}\left[\Lambda_5(t)-\frac{1}{2\pi}\int_0^\infty Q_4(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\eta}\,\mathrm{d}\eta\right.$$

$$+\frac{1}{2\pi}\int_0^\infty Q_4(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta$$

$$\left.\left.-\frac{1}{2\pi l_1^2}\int_0^\infty Q_4(\eta,t)\frac{\mathrm{e}^{-\eta t}\ \sin[x\mathcal{P}(\eta)]}{\eta\sqrt{(\eta+\chi)^2+\xi}}\,\mathrm{d}\eta\right]\right\}, \tag{3.5}$$

$$u_3 = H(t-cx)\left\{\frac{G}{\lambda^2P^2l_2^2}\left[\Lambda_5(t-cx)\exp\left(\frac{-cx}{2\lambda}\right)+cx\int_{cx}^t\Lambda_5(t-\zeta)B(x,\zeta)\,\mathrm{d}\zeta\right]\right\}, \tag{3.6}$$

where

$$\mathcal{P}(\eta)=\frac{l_1}{l_2}\sqrt{\frac{\eta-l_1^{-2}+\sqrt{(\eta+\chi)^2+\xi}}{2}},$$

$$B(x,\zeta)=\frac{\mathrm{e}^{-\zeta/(2\lambda)}}{2\lambda\sqrt{\zeta^2-\lambda Px^2}}I_1\left(\frac{\sqrt{\zeta^2-\lambda Px^2}}{2\lambda}\right), \tag{3.7}$$

$$Q_\ell(\eta,t)=\int_0^t \mathrm{e}^{\eta\zeta}\Lambda_\ell(\zeta)\,\mathrm{d}\zeta \quad (\ell=3,4),\ \Lambda_m(t)=\mathcal{L}^{-1}\left[\frac{1}{s^{m-3}\Pi(s)}\right] \quad (m=3,4,5), \quad (3.8)$$

and

$$Q_3(\eta,t)=\begin{cases}\begin{cases}\dfrac{\mathrm{e}^{\eta_1 t}}{\eta_1(n_3-n_1)(n_2-n_1)}-\dfrac{\mathrm{e}^{\eta_2 t}}{\eta_2(n_2-n_1)(n_3-n_2)} \\ \quad +\dfrac{\mathrm{e}^{\eta_3 t}}{\eta_3(n_3-n_2)(n_3-n_1)}-\dfrac{1}{\eta_1\eta_2\eta_3}, & \Delta<0, \\ \dfrac{\mathrm{e}^{\eta_0 t}(2-2\eta_0 t+\eta_0^2t^2)-2}{2\eta_0^3}, & \Delta=0^*, \\ \dfrac{\eta_0-n_1+n_0}{n_0^2(n_1-n_0)^2} \\ \quad -\dfrac{\mathrm{e}^{\eta_0 t}(\eta_0-n_1+n_0+\eta_0 t(n-1-n_0))}{n_0^2(n_1-n_0)^2} \\ \quad +\dfrac{\mathrm{e}^{\eta_1 t}-1}{\eta_1(n_1-n_0)^2}, & \Delta=0^{**}, \\ \dfrac{b\eta_1(a-n_1+\eta_a)(1-\mathrm{e}^{\eta_a t}\ \cos[bt])}{b\eta_1(b^2+\eta_a^2)(b^2+(a-n_1)^2)} \\ \quad +\dfrac{b(b^2+\eta_a^2)(\mathrm{e}^{\eta_1 t}-1)}{b\eta_1(b^2+\eta_a^2)(b^2+(a-n_1)^2)} \\ \quad -\dfrac{\eta_1\,\mathrm{e}^{\eta_a t}(b^2+\eta_a n_1-a\eta_a)\sin[bt]}{b\eta_1(b^2+\eta_a^2)(b^2+(a-n_1)^2)}, & \Delta>0,\end{cases} & P\neq 1; \\ \begin{cases}\dfrac{\mathrm{e}^{\eta_4 t}-1}{\eta_4 n_4^2-\eta_4 n_4 n_5}+\dfrac{\mathrm{e}^{\eta_5 t}-1}{\eta_5 n_5^2-\eta_5 n_4 n_5} \\ \quad +\dfrac{\mathrm{e}^{\eta t}-1}{\eta n_4 n_5}, \quad \lambda\neq l_2^2-l_1^2, \\ \dfrac{\eta^2\,\mathrm{e}^{n_6 t}-\eta^2+\eta n_6-\eta_6 n_6}{\eta^2\eta_6 n_6^2} \\ \quad -\dfrac{\mathrm{e}^{\eta t}(\eta n_6 t+\eta-n_6)}{\eta^2 n_6^2}, \quad \lambda=l_2^2-l_1^2,\end{cases} & P=1;\end{cases} \quad (3.9)$$

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Q_4(\eta,t)=\begin{cases}
\begin{cases}
\left[\dfrac{e^{\eta_1 v}}{\eta_1 n_1(n_2-n_1)(n_3-n_1)}\right. \\[2ex]
\quad -\dfrac{e^{\eta_2 v}}{\eta_2 n_2(n_2-n_1)(n_3-n_2)}-\dfrac{e^{\eta v}}{\eta n_1 n_2 n_3} \\[2ex]
\quad \left.+\dfrac{e^{\eta_3 v}}{\eta_3 n_3(n_3-n_1)(n_3-n_2)}\right]_0^t, & \Delta<0, \\[2ex]
\left[\dfrac{e^{\eta_0 v}(\eta_0^2 n_0^2 v^2-2v(\eta_0^2 n_0+\eta_0 n_0^2)}{2\eta_0^3 n_0^3}\right. \\[2ex]
\quad \left.\dfrac{+2n_0^2+2\eta_0 n_0+2n_0^2)-2\eta_0^3 v}{2\eta_0^3 n_0^3}\right]_0^t, & \Delta=0^{*}, \\[2ex]
\left[\dfrac{e^{\eta_1 v}}{\eta_1 n_1(n_1-n_0)^2}-\dfrac{e^{\eta v}}{\eta n_0^2 n_1}\right. \\[2ex]
\quad \left.+\dfrac{e^{\eta_0 v}(v(\eta_0 n_0^2-\eta_0 n_0 n_1)-2\eta_0 n_0-n_0^2+\eta_0 n_1+n_0 n_1)}{\eta_0^2 n_0^2(n_1-n_0)^2}\right]_0^t, & \Delta=0^{**}, \\[2ex]
\left[\dfrac{e^{\eta_1 v}}{\eta_1 n_1(a^2+b^2-2an_1+n_1^2)}-\dfrac{e^{\eta_1 v}}{\eta n_1(a^2+b^2)}\right. \\[2ex]
+\dfrac{e^{\eta_a v}(b(b^2-2a\eta_a+an_1+\eta_a n_1-a^2)\cos[bv]}{b(a^2+b^2)(\eta_a^2+b^2)(a^2+b^2-2an_1+n_1^2)} \\[2ex]
\left.+\dfrac{(a^2\eta_a-2ab^2-b^2\eta_a+b^2 n_1-a\eta_a n_1)\sin[bv])}{b(a^2+b^2)(\eta_a^2+b^2)(a^2+b^2-2an_1+n_1^2)}\right]_0^t, & \Delta>0,
\end{cases} & P\neq 1; \\[4ex]
\begin{cases}
\left[\dfrac{e^{\eta_5 v}}{\eta_5 n_5^2(n_5-n_4)}-\dfrac{e^{\eta_4 v}}{\eta_4 n_4^2(n_5-n_4)}\right. \\[2ex]
\quad \left.+\dfrac{e^{\eta v}(\eta n_4 n_5 v+\eta n_4-n_4 n_5+\eta n_5)}{\eta^2 n_4^2 n_5^2}\right]_0^t, & \lambda\neq l_2^2-l_1^2, \\[2ex]
\left[\dfrac{e^{\eta_6 v}}{\eta_6 n_6^3}-\dfrac{e^{\eta v}(2\eta^2-2\eta n_6+2n_6^2}{2\eta^3 n_6^3}\right. \\[2ex]
\quad \left.\dfrac{+2\eta n_6 v(n-n_6)+\eta^2 n_6^2 v^2)}{2\eta^3 n_6^3}\right]_0^t, & \lambda=l_2^2-l_1^2,
\end{cases} & P=1;
\end{cases}
\tag{3.10}
$$

$$\Lambda_3(t) = \begin{cases} \begin{cases} \dfrac{(n_3-n_2)\,e^{n_1 t} - (n_3-n_1)\,e^{n_2 t} + (n_2-n_1)\,e^{n_3 t}}{(n_2-n_1)(n_3-n_2)(n_3-n_1)}, & \Delta < 0, \\[2ex] \dfrac{t^2\,e^{n_0 t}}{2}, & \Delta = 0^{*}, \\[2ex] \dfrac{e^{n_1 t} - (n_1-n_0)t\,e^{n_0 t} - e^{n_0 t}}{(n_1-n_0)^2}, & \Delta = 0^{**}, \\[2ex] \dfrac{b\,e^{n_1 t} + e^{at}(a\ \sin[bt] - n_1\ \sin[bt] - b\ \cos[bt])}{b(a^2+b^2-2an_1+n_1^2)}, & \Delta > 0, \end{cases} & P \neq 1; \\[4ex] \begin{cases} \dfrac{n_4(e^{-n_5 t}-1) - n_5(e^{-n_4 t}-1)}{n_4 n_5^2 - n_5 n_4^2}, & \lambda \neq l_2^2 - l_1^2, \\[2ex] \dfrac{e^{\eta_6 t} - n_6 t - 1}{n_6^2}, & \lambda = l_2^2 - l_1^2, \end{cases} & P = 1; \end{cases} \tag{3.11}$$

$$\Lambda_4(t) = \begin{cases} \begin{cases} \dfrac{e^{n_1 t}}{n_1(n_2-n_1)(n_3-n_1)} - \dfrac{e^{n_2 t}}{n_2(n_2-n_1)(n_3-n_2)} \\ \quad + \dfrac{e^{n_3 t}}{n_3(n_3-n_1)(n_3-n_2)} - \dfrac{1}{n_1 n_2 n_3}, & \Delta < 0, \\[2ex] \dfrac{e^{n_0 t}(n_0^2 t^2 - 2n_0 t + 2) - 2}{2n_0^3}, & \Delta = 0^{*}, \\[2ex] \dfrac{e^{n_0 t}(n_0^2 t - n_0 n_1 t + n_1 - 2n_0)}{n_0^2(n_1-n_0)^2} + \dfrac{e^{n_1 t}}{n_1(n_1-n_0)^2} - \dfrac{1}{n_0^2 n_1}, & \Delta = 0^{**}, \\[2ex] \dfrac{b\,e^{n_1 t}(a^2+b^2) - b(n_1^2 - 2an_1 + a^2) + b^2)}{bn_1(a^2+b^2-2an_1+n_1^2)} \\ + \dfrac{n_1\,e^{nt}(b(n_1-2a)\ \cos[bt] + (a^2 - an_1 - b^2)\ \sin[bt])}{bn_1(a^2+b^2-2an_1+n_1^2)}, & \Delta > 0, \end{cases} & P \neq 1; \\[4ex] \begin{cases} \dfrac{t(n_4 n_5^2 - n_4^2 n_5) - n_4^2 + n_5^2 + n_4^2\,e^{n_5 t} - n_5^2\,e^{n_4 t}}{n_4^2 n_5^2(n_5-n_4)}, & \lambda \neq l_2^2 - l_1^2, \\[2ex] \dfrac{2\,e^{n_6 t} - 2 - 2n_6 t - n_6^2 t^2}{2n_6^3}, & \lambda = l_2^2 - l_1^2, \end{cases} & P = 1; \end{cases} \tag{3.12}$$

$$\Lambda_5(t) = \begin{cases} \begin{cases} \left[\dfrac{e^{n_1\zeta}}{n_1^2(n_2-n_1)(n_3-n_1)} - \dfrac{e^{n_2\zeta}}{n_2^2(n_2-n_1)(n_3-n_2)} \right. \\ \left. + \dfrac{e^{n_3\zeta}}{n_3^2(n_3-n_1)(n_3-n_2)} - \dfrac{\zeta}{n_1 n_2 n_3}\right]_0^t, & \Delta < 0, \\ \left[\dfrac{e^{n_0\zeta}(n_0^2\zeta^2 - 4n_0\zeta + 6) - 2n_0\zeta)}{2n_0^4}\right]_0^t, & \Delta = 0^*, \\ \left[\dfrac{e^{n_0\zeta}(n_0^2\zeta - n_0 n_1\zeta + 2n_1 - 3n_0)}{n_0^3(n_1-n_0)^2} \right. \\ \left. + \dfrac{e^{n_1\zeta}}{n_1^2(n_1-n_0)^2} - \dfrac{\zeta}{n_0^2 n_1}\right]_0^t, & \Delta = 0^{**}, \qquad P \neq 1; \\ \left[\dfrac{e^{n_1\zeta}(a^4+2a^2b^2+b^4)}{n_1^2(a^2+b^2)^2(a^2+b^2-2an_1+n_1^2)} \right. \\ - \dfrac{n_1\zeta(a^4+2a^2b^2-b^4-2a^3n_1-2ab^2n_1+a^2n_1^2+b^2n_1^2)}{n_1^2(a^2+b^2)^2(a^2+b^2-2an_1+n_1^2)} \\ + \dfrac{e^{a\zeta}(b(b^2-3a^2+2an_1)\cos[b\zeta]}{b(a^2+b^2)^2(a^2+b^2-2an_1+n_1^2)} \\ \left. + \dfrac{(a^3-3ab^2-a^2n_1+b^2n_1)\sin[b\zeta])}{b(a^2+b^2)^2(a^2+b^2-2an_1+n_1^2)}\right]_0^t, & \Delta > 0, \end{cases} \\ \begin{cases} \left[\dfrac{\zeta^3}{6n_4n_5} - \dfrac{e^{n_4\zeta}}{n_4^4(n_5-n_4)} \right. \\ \left. + \dfrac{e^{n_5\zeta}}{n_5^4(n_5-n_4)} + \dfrac{(n_4+n_5)\zeta^2}{2n_4^2n_5^2}\right]_0^t, \quad \lambda \neq l_2^2 - l_1^2, \\ \left[\dfrac{6\,e^{n_6\zeta} - 6n_6\zeta - 3n_6^2\zeta^2 - n_6^3\zeta^3}{6n_6^4}\right]_0^t, \quad \lambda = l_2^2 - l_1^2, \end{cases} \qquad P = 1; \end{cases} \tag{3.13}$$

and where $I_1(\cdot)$ is the modified Bessel function of the first kind of order one; $\mathcal{L}^{-1}[\cdot]$ denotes the inverse Laplace transform operator;

$$\Delta = \frac{(2l_1^6 - 3P_1^4l_2^2 + 9\lambda Pl_1^2l_2^2 - 27\lambda Pl_2^4 - 3P^2l_1^2l_2^4 + 9\lambda P^2l_2^4 + 2P^3l_2^6)^2}{2916(\lambda Pl_2^2)^6}$$
$$- \frac{4(l_2^4P^2 - l_1^2l_2^2P + 3\lambda Pl_2^2 + l_1^4)^3}{2916(\lambda Pl_2^2)^6} \tag{3.14}$$

is the discriminant of the cubic polynomial $\Pi(s)$; and $n_1, n_2, n_3 \neq 0$ are the possible roots of $\Pi(s)$ for $P \neq 1$. For $\Delta < 0$ all roots are real and distinct, for $\Delta = 0$ all roots are real and we have (*) $n_1 = n_2 = n_3 \equiv n_0$ or (**) $n_1 \neq n_2 = n_3 \equiv n_0$, and for $\Delta > 0$ one root, say n_1, is real and the other two form the complex conjugate pair $n_{2,3} = a \pm ib (b \neq 0)$. For $P = 1$ all roots of $\Pi(s)$ are real and we denote the possible nonzero roots by n_4, n_5, and n_6, where

$$n_{4,5} = \frac{-(2l_2^2 - l_1^2) \pm \sqrt{l_1^4 + 4\lambda l_2^2}}{2\lambda l_2^2}, \qquad n_6 = -\frac{l_2^2 + \lambda}{\lambda l_2^2}. \tag{3.15}$$

Finally, we let $\eta_\ell = n_\ell + \eta (\ell = 0, 1, 2, 3, 4, 5, 6)$ and $\eta_a = a + \eta$.

Clearly, there are six possible cases of $u(x, t)$, four corresponding to $P \neq 1$ and the other two for the case $P = 1$. Furthermore, we call attention to the simple relationship between λ and the dipolar constants which determines the character of the solution for the $P = 1$ case. Finally, we observe that u_3 is a propagating wave with a phase velocity of $1/c > 0$; it is the only propagating term of the velocity field.

4. Propagating discontinuities

In this article we will only investigate u and its derivatives for propagating discontinuities. The behavior of θ in this regard has been discussed in [10] and will not be examined here. Clearly, the parts of the velocity distribution denoted by u_1 and u_2, and their derivatives, remain continuous everywhere for $t > 0$. Thus we need only examine u_3 and its derivatives for jumps. In this work, we use the method developed by Boley [33] for determining the propagating discontinuities of a function from its transform (see also Chadwick and Powdrill [34]). We will apply Boley's method to $u_3(x, t)$ and its temporal derivatives in the transform domain. Then Hadamard's lemma [35, pp. 491–525]

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathcal{S}\left[\frac{\partial^{n+q} u}{\partial x^q \partial t^n}\right] = \mathcal{S}\left[\frac{\partial^{n+q+1} u}{\partial x^q \partial t^{n+1}}\right] + \frac{1}{c}\mathcal{S}\left[\frac{\partial^{n+q+1} u}{\partial x^{q+1} \partial t^n}\right], \tag{4.1}$$

where $\mathcal{S}[\cdot]$ denotes the jump discontinuity (or saltus) of a function across a singular surface (or wavefront), the operator $\mathrm{D}/\mathrm{D}t$ denotes differentiation with respect to time following the wavefront, and n and q are nonnegative integers, will be used with the results obtained from Boley's method to determine the jumps in the spatial and mixed derivatives of u. To this end we employ the well-known properties of the Laplace transform to obtain the relation

$$\overline{\frac{\partial^n u_3(x, t)}{\partial t^n}} = s^n \bar{u}_3(x, s). \tag{4.2}$$

Substituting Equation (2.27) in Equation (4.2) and expanding the result for large s (*i.e.*, small-time), we have

$$\overline{\frac{\partial^n u_3(x, t)}{\partial t^n}} = \frac{G\bar{f}(s)\,\mathrm{e}^{-xs\sqrt{\lambda P} - x\sqrt{P/4\lambda}}}{\lambda^2 P^2 l_2^2}\left[\frac{1}{s^{4-n}} + \frac{1}{s^{5-n}}\left(\frac{x\sqrt{\lambda P}}{8\lambda^2} - \frac{2Pl_2^2 - l_1^2}{\lambda P l_2^2}\right) + \cdots\right]. \tag{4.3}$$

Applying the method of Boley [33] to Equation (4.3) we find, for $n = 0, 1, 2, 3$ and $f(t) = \delta(t)$,

$$\mathcal{S}[u] = \mathcal{S}[u_t] = \mathcal{S}[u_{tt}] = 0,\ \mathcal{S}[u_{ttt}] = \frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{(\lambda P l_2)^2}. \tag{4.4}$$

Table 1. Propagating jumps in u resulting from $f(t) = \delta(t)$

$f(t)$	$\mathcal{S}[u_{xxx}]$	$\mathcal{S}[u_{xxt}]$	$\mathcal{S}[u_{xtt}]$	$\mathcal{S}[u_{ttt}]$
$\delta(t)$	$\frac{-G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{l_2^2\sqrt{\lambda P}}$	$\frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{\lambda P l_2^2}$	$\frac{-G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{\lambda P l_2^2\sqrt{\lambda P}}$	$\frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{(\lambda P l_2)^2}$

Table 2. Propagating jumps in u resulting from $f(t) = H(t)$

$f(t)$	$\mathcal{S}[u_{xxxx}]$	$\mathcal{S}[u_{xxxt}]$	$\mathcal{S}[u_{xxtt}]$	$\mathcal{S}[u_{xttt}]$	$\mathcal{S}[u_{tttt}]$
$H(t)$	$\frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{l_2^2}$	$\frac{-G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{l_2^2\sqrt{\lambda P}}$	$\frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{\lambda P l_2^2}$	$\frac{-G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{\lambda P l_2^2\sqrt{\lambda P}}$	$\frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{(\lambda P l_2)^2}$

Hence, at the wavefront, u, u_t, and u_{tt} are all continuous and all time derivatives of order greater than three do not exist. From Hadamard's lemma we find, using (4.4), that for all n and q such that $n + q \leqslant 2$

$$\mathcal{S}\left[\frac{\partial^{n+q} u}{\partial x^q \partial t^n}\right] \equiv 0. \tag{4.5}$$

Thus (4.1) reduces to

$$\mathcal{S}\left[\frac{\partial^{n+q+1} u}{\partial x^{q+1} \partial t^n}\right] = -c\mathcal{S}\left[\frac{\partial^{n+q+1} u}{\partial x^q \partial t^{n+1}}\right], \tag{4.6}$$

for all n and q such that $n + q \leqslant 2$. Hence, we are able to determine the propagating jump discontinuities in u and all its derivatives across the wavefront $x = t/c$ for $f(t) = \delta(t)$. Taking $f(t) = H(t)$ and $n = 0, 1, 2, 3, 4$ we arrive at

$$\mathcal{S}[u] = \mathcal{S}[u_t] = \mathcal{S}[u_{tt}] = \mathcal{S}[u_{ttt}] = 0, \mathcal{S}[u_{tttt}] = \frac{G\,\mathrm{e}^{-x\sqrt{P/4\lambda}}}{(\lambda P l_2)^2}. \tag{4.7}$$

In this case we find that, at the wavefront, u, u_t, u_{tt}, and u_{ttt} are all continuous and all time derivatives of order greater than four do not exist. Thus, in a manner similar to that shown above for $f(t) = \delta(t)$, the propagating jump discontinuities in u and all its derivatives can be determined for $f(t) = H(t)$. Tables 1 and 2 give the nonzero finite jumps in u and its derivatives for $f(t) = \delta(t)$ and $f(t) = H(t)$, respectively. For a partial differential equation of order N, discontinuities in u itself and its partial derivatives of orders $1, \ldots, N - 1$ are said to be strong; discontinuities in partial derivatives of order $\geqslant N$ are known as weak [36]. Thus, as was found for a viscous Newtonian fluid [10], impulsive temperature boundary data produces strong discontinuities in the velocity while Heaviside-type temperature boundary data gives rise to weak discontinuities in u. In addition, it can be seen from Tables 1 and 2 that the magnitude of each of the jumps is proportional to the constant G/l_2^2. Hence, l_2 is the only dipolar constant to have an impact on the magnitudes of these discontinuities and the coupling constant G is the only parameter to influence their algebraic signs. Furthermore, we observe that letting $\lambda \to 0$ (*i.e.*, using Fourier's heat law) results in all jumps given in Tables 1 and 2 going to zero. Actually, it can be easily shown using Boley's method that u is of class $C^\infty(t > 0)$ under Fourier's heat law or if thermal effects are removed (*i.e.*, $G = 0$). Finally we note that, although somewhat more laborious, the method of Boley [33] alone could have been used to determine the jumps in u and all its derivatives (see [10]).

5. Monopolar and dipolar stresses

In dimensional form, the nonzero components of the monopolar and dipolar stresses, σ_{ij} and Σ_{ijk} respectively, are given by [25]

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p = -\Phi(t), \qquad \sigma_{zx} = \mu \frac{\partial u}{\partial x}, \qquad \sigma_{xz} = -(h_1 + h_3)\frac{\partial^3 u}{\partial x^3} + \mu \frac{\partial u}{\partial x},$$

$$\Sigma_{zzz} = -2\Psi(t) + h_1 \frac{\partial^2 u}{\partial x^2}, \qquad \Sigma_{xxz} = (h_1 + h_3)\frac{\partial^2 u}{\partial x^2}, \qquad \Sigma_{yyz} = h_1 \frac{\partial^2 u}{\partial x^2},$$

$$\Sigma_{zxx} = \Sigma_{xzx} = -\Psi(t) + h_2 \frac{\partial^2 u}{\partial x^2}, \qquad \Sigma_{yzy} = \Sigma_{zyy} = -\Psi(t). \tag{5.1}$$

We observe that, with the possible exception of those components which depend on the arbitrary functions Ψ and Φ, all stress components are of class $C^\infty(t > 0)$ when Fourier's heat law is assumed or if thermal effects are removed. As shown by Jordan [29, p. 29], when the MCF model is considered and $f(t) = \delta(t)$, the monopolar stress component σ_{xz} suffers a jump discontinuity (see Table 1). In nondimensional form, this jump is given by

$$\mathcal{J}[\sigma_{xz}] = \frac{G}{\sqrt{\lambda P}}\, \mathrm{e}^{-x\sqrt{P/4\lambda}}. \tag{5.2}$$

For $f(t) = H(t)$, Table 2 indicates that jump discontinuities now occur in the first derivatives of σ_{xz}. We express these jumps as

$$\mathcal{J}\left[\frac{\partial \sigma_{xz}}{\partial x}\right] = -G\, \mathrm{e}^{-x\sqrt{P/4\lambda}}, \qquad \mathcal{J}\left[\frac{\partial \sigma_{xz}}{\partial t}\right] = \frac{G}{\sqrt{\lambda P}}\, \mathrm{e}^{-x\sqrt{P/4\lambda}}. \tag{5.3}$$

Lastly, it is of interest to note that both Equations (5.2) and (5.3) are independent of the dipolar constants l_1 and l_2.

6. Special and limiting cases

Returning to Equations (2.25)–(2.27), we observe that for dipolar fluids with large Prandtl numbers (*e.g.*, various types of oils) all terms containing the coupling constant G are small. Thus, as P becomes large, $u(x,t) \to u_1(x,t)$, where u_1 is the solution to (2.17) for $G = 0$.

Following Guram [25], we take $l_1 = l_2$. Doing so, we find that the transform domain solution reduces to

$$\bar{u}(x,s) \;=\; \frac{\mathrm{e}^{-x/L}}{s - L^{-2}}\left[s\bar{g}(s) - \frac{M_1}{s} + \frac{G\bar{f}(s)}{\lambda P L^2 (s-m_1)(s-m_2)}\right]$$

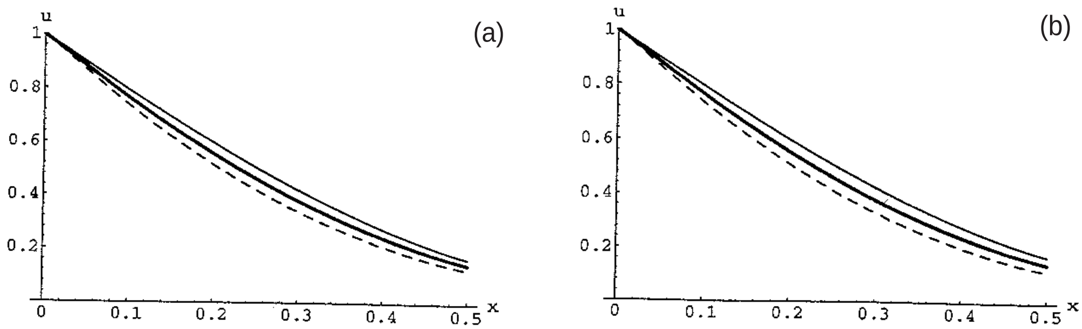

Figure 1. u *vs.* x for $L = 0{\cdot}1$, $M = 1{\cdot}0$, $P = 0{\cdot}7$, $t = 0{\cdot}05$. Bold: $G = 0$, solid: $G = 5{\cdot}0$, broken: $G = -5{\cdot}0$. (a) $\lambda = 0{\cdot}01$, (b) $\lambda = 0$.

$$+\frac{-\mathrm{e}^{-x\sqrt{s}}}{s - L^{-2}}\left[\frac{\bar{g}(s)}{L^2} - \frac{M_1}{s} + \frac{G\bar{f}(s)}{\lambda P L^2 s(s - m_3)}\right]$$

$$+\frac{G\bar{f}(s)\,\mathrm{e}^{-cx\sqrt{(s+h)^2-h^2}}}{\lambda^2 P^2 L^2 s(s - m_1)(s - m_2)(s - m_3)}, \tag{6.1}$$

where $L \equiv l_1 = l_2$, $h = (2\lambda)^{-1}$,

$$m_1 = -\frac{1}{2\lambda} - \frac{\sqrt{4\lambda + PL^2}}{2\lambda L\sqrt{P}}, \qquad m_2 = -\frac{1}{2\lambda} + \frac{\sqrt{4\lambda + PL^2}}{2\lambda L\sqrt{P}}, \qquad m_3 = \frac{1 - P}{\lambda P}. \tag{6.2}$$

Equation (6.1) can be easily inverted and allows us to obtain closed-form solutions for the velocity field. These solutions are given for $f(t) = g(t) = H(t)$ in Appendix A. Clearly when L is small, the behavior of the dipolar fluid considered here is approximately that of the viscous Newtonian fluid studied by Puri and Kythe [10]. However, when ν, the kinematic viscosity, is small, then L can become large. From (6.1) we observe that for large L, all terms containing the coupling constant G can be neglected. Thus, thermal effects on $u(x, t)$ again disappear. Hence, when L or P is large the equation of motion, Equation (2.17), becomes uncoupled from Equation (2.18), the heat-conduction equation.

Lastly, we note that the velocity field under the classical (Fourier's) heat law can be found by letting $\lambda \to 0$ in (2.25)–(2.27) and then inverting, except for the singular case of $l_1 = l_2$ and $P = 1$ simultaneously. However, to limit the size of this article, we will only give expressions for u based on Fourier's heat law for the special case of equal dipolar constants and $f(t) = g(t) = H(t)$. These solutions are presented in Appendix B.

7. Numerical results

The solution corresponding to $f(t) = g(t) = \delta(t)$ is the fundamental solution, with respect to time, of Equation (2.17) and is, therefore, a basic result of theoretical importance. Here, however, we will discuss numerical results for the more physically applicable case of $f(t) = g(t) = H(t)$. Figures 1–3 depict u *vs.* x and are generated for the case $L \equiv l_1 = l_2$. With a Prandtl number of 0·7, the velocity profiles shown in Figures 1(a) and 1(b) correspond to gases like air or helium. They illustrate the effects of the coupling constant G on u under the MCF

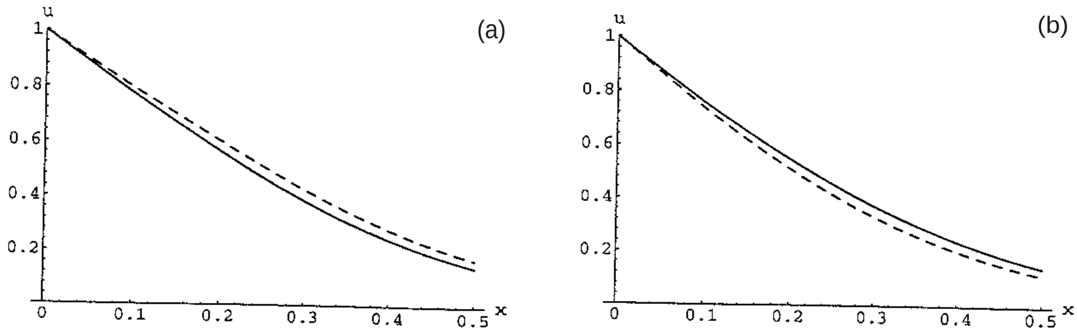

Figure 2. u *vs.* x for $L = 0{\cdot}1$, $M = 1{\cdot}0$, $P = 0{\cdot}7$, $t = 0{\cdot}05$. Solid: $\lambda = 0{\cdot}2$, broken: $\lambda = 0$. (a) $G = 5{\cdot}0$, (b) $G = -5{\cdot}0$.

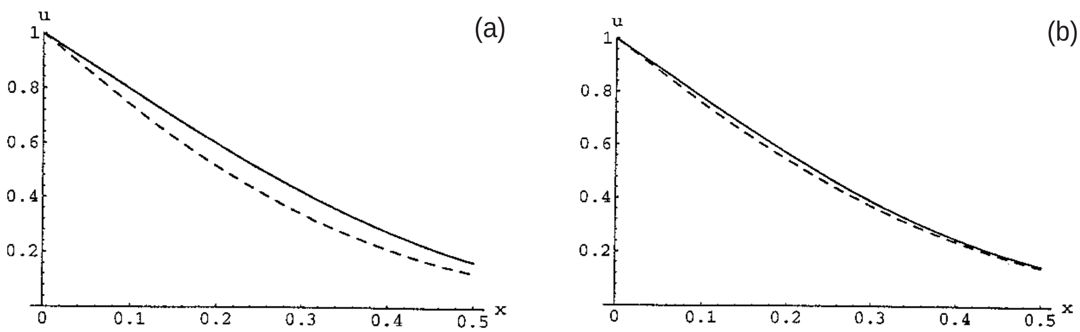

Figure 3. u *vs.* x for $\lambda = 0{\cdot}01$, $L = 0{\cdot}1$, $M = 1{\cdot}0$, $t = 0{\cdot}05$. Solid: $G = 5{\cdot}0$, broken: $G = -5{\cdot}0$. (a) $P = 0{\cdot}7$, (b) $P = 3{\cdot}7$.

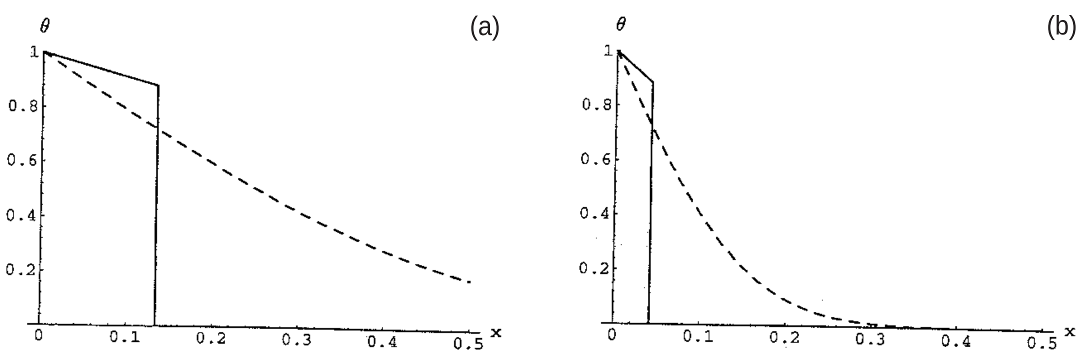

Figure 4. θ *vs.* x for $t = 0{\cdot}05$. Solid: $\lambda = 0{\cdot}2$, broken: $\lambda = 0$. (a) $P = 0{\cdot}7$, (b) $P = 7{\cdot}0$.

model ($\lambda > 0$) and Fourier's heat law ($\lambda = 0$), respectively. Clearly, increasing G increases the velocity for both the $\lambda > 0$ and $\lambda = 0$ cases. In Figure 2 we see that an increase in λ appears to decrease velocity when the fluid is heated ($G > 0$) and increases velocity when the fluid is cooled ($G < 0$). It is of interest to note that the results illustrated in Figures 1 and 2 were also found to be valid for viscous Newtonian fluids [10]. Figures 3(a) and 3(b) show the effects of P on u under both heating (solid curve) and cooling (broken curve) conditions. It is obvious that for both $P = 0{\cdot}7$ and $P = 3{\cdot}7$ ($P = 3{\cdot}7$ corresponding to a freon-type fluid), $G > 0$ results in a greater velocity. Furthermore, we observe that as P increase, the $G > 0$ curve drops while the $G < 0$ curve rises (i.e., each tending towards the curve corresponding to $G = 0$). Thus, as was shown in Section 7, we find that increasing P reduces the influence of temperature on the velocity field.

Finally, in Figure 4 we plot θ *vs.* x for both the MCF model (solid curve) and Fourier's (broken curve) heat law. In Figure 4(a) we again use a Prandtl number of 0·7, while in Figure 4(b) a Prandtl number of 7·0, corresponding to water, is employed. The discontinuity in θ under the MCF model is clearly visible in Figure 4. Also, we note that the temperature based on the MCF model is greater than the temperature resulting from Fourier's heat law in the region where the MCF based temperature wave has propagated (*i.e.*, the interval $0 < x < t/c$). Furthermore, this result seems to be independent of P. The reason for this is that the solution of the parabolic equation resulting from Fourier's heat law instantaneously diffuses the heat applied to the boundary $x = 0$ throughout the entire half-space $x > 0$. In contrast, the MCF model gives rise to a hyperbolic equation. As is generally known, the solution of a hyperbolic equation propagates boundary data into the solution domain at a finite speed (in our case $1/c$). Thus, for a given $t > 0$, heat supplied at $x = 0$ is restricted to a slab of thickness t/c (see also [4, pp. 172–177]).

8. Conclusions

Based on the analysis presented above, we give the following conclusions:

- (i) Under the MCF model there are six possible cases of $u(x, t)$, four corresponding to $P \neq 1$ (the cases of the discriminant Δ) and two corresponding to $P = 1$.
- (ii) As L or P becomes large, Equations (2.17) and (2.18) uncouple; thermal influences on u vanish.
- (iii) A discontinuity in velocity boundary data does not propagate.
- (iv) For $\lambda > 0$, impulsive temperature boundary data produces jumps in the third order derivatives of u and Heaviside-type temperature boundary data produces jumps in the fourth order derivatives of u. Since the equation of motion is of fourth order, taking $f(t) = \delta(t)$ produces strong discontinuities in u while discontinuities resulting from $f(t) = H(t)$ are weak.
- (v) The magnitudes of the jumps in u are proportional to G/l_2^2 and are independent of the dipolar constant l_1. The magnitudes of the jumps in the monopolar stress component σ_{xz} are proportional to G and are independent of both l_1 and l_2.
- (vi) Increasing G increases velocity.
- (vii) Increasing λ or P reduces velocity for $G > 0$ (heating) and increases velocity for $G < 0$ (cooling).
- (viii) The temperature resulting from the MCF model is greater than the temperature satisfying Fourier's heat law on the interval $0 < x < t/c$.

Appendix A

Inverting Equation (6.1) for $f(t) = H(t)$ and $P \neq 1$, we obtain

$$\begin{aligned}
u(x,t) \;=\; & H(t)\Bigg\{ \mathrm{e}^{-x/L}\Bigg[1 + (\mathrm{e}^{t/L^2} - 1)(1 - M_1 L^2) + \frac{G}{\lambda P}\Bigg(\frac{L^4\,\mathrm{e}^{t/L^2}}{(1 - m_1L^2)(1 - m_2L^2)} \\
& - \frac{\mathrm{e}^{m_1 t}}{m_1(1 - m_1L^2)(m_1 - m_2)} + \frac{\mathrm{e}^{m_2 t}}{m_2(1 - m_2L^2)(m_1 - m_2)} - \frac{1}{m_1 m_2}\Bigg)\Bigg] \\
& + \frac{\lambda P(1 - M_1L^2)(1 - m_3L^2) + GL^4}{\lambda P(1 - m_3L^2)} \\
& \times \Bigg[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \frac{\mathrm{e}^{t/L^2}}{2}(\mathrm{e}^{x/L}\operatorname{erfc}[v_+(x/L, tL^2)] + \mathrm{e}^{-x/L}\operatorname{erfc}[v_-(x/L, t/L^2)])\Bigg] \\
& + \frac{G}{1 - P}\left[x\sqrt{\frac{t}{\pi}}\,\mathrm{e}^{-x^2/4t} - \left(\frac{x^2}{2} + t\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)\right] \\
& - \frac{G}{\lambda P m_3^2(1 - m_3L^2)}\Bigg[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \frac{\mathrm{e}^{m_3 t}}{2}(\mathrm{e}^{x\sqrt{m_3}}\operatorname{erfc}[v_+(x\sqrt{m_3}, tm_3)] \\
& + \mathrm{e}^{-x\sqrt{m_3}}\operatorname{erfc}[v_-(x\sqrt{m_3}, tm_3)])\Bigg]\Bigg\} \\
& + H(t - cx)\Bigg\{\frac{G}{\lambda^2 P^2 L^2}\Bigg[cx\int_{cx}^{t}\Bigg(\frac{\mathrm{e}^{m_1(t-\zeta)}}{m_1^2(m_1 - m_2)(m_1 - m_3)} \\
& + \frac{\mathrm{e}^{m_2(t-\zeta)}}{m_2^2(m_2 - m_1)(m_2 - m_3)} + \frac{\mathrm{e}^{m_3(t-\zeta)}}{m_3^2(m_3 - m_1)(m_3 - m_2)} \\
& - \frac{m_1m_2 + m_1m_3 + m_2m_3}{m_1^2 m_2^2 m_3^2} - \frac{t - \zeta}{m_1 m_2 m_3}\Bigg)B(x,\zeta)\,\mathrm{d}\zeta \\
& + \mathrm{e}^{-c^h x}\Bigg(\frac{\mathrm{e}^{m_1(t-cx)}}{m_1^2(m_1 - m_2)(m_1 - m_3)} \\
& + \frac{\mathrm{e}^{m_2(t-cx)}}{m_2^2(m_2 - m_1)(m_2 - m_3)} + \frac{\mathrm{e}^{m_3(t-cx)}}{m_3^2(m_3 - m_1)(m_3 - m_2)} \\
& - \frac{m_1m_2 + m_1m_3 + m_2m_3}{m_1^2 m_2^2 m_3^2} - \frac{t - cx}{m_1 m_2 m_3}\Bigg)\Bigg]\Bigg\},
\end{aligned} \tag{A.1}$$

where erfc $[\cdot]$ is the complementary error function and

$$v_\pm(x,t) = \frac{x}{2\sqrt{t}} \pm \sqrt{t}. \tag{A.2}$$

For $f(t) = H(t)$ and $P = 1$, Equation (6.1) becomes after inversion

$$
\begin{aligned}
u(x,t) &= H(t)\Bigg\{ e^{-x/L}\Bigg[1+(e^{t/L^2}-1)(1-M_1L^2)+\frac{G}{\lambda}\Bigg(\frac{L^4 e^{t/L^2}}{(1-m_1L^2)(1-m_2L^2)} \\
&\quad -\frac{e^{m_1 t}}{m_1(1-m_1L^2)(m_1-m_2)}+\frac{e^{m_2 t}}{m_2(1-m_2L^2)(m_1-m_2)}-\frac{1}{m_1m_2}\Bigg)\Bigg] \\
&\quad +\frac{\lambda(M_1L^2-1)-GL^4}{\lambda}\Bigg[\frac{e^{t/L^2}}{2}(e^{x/L}\operatorname{erfc}[v_+(x/L,t/L^2)] \\
&\quad + e^{-x/L}\operatorname{erfc}[v_-(x/L,t/L^2)]) \\
&\quad -\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)\Bigg]+\frac{G}{\lambda}\Bigg[L^2t+\left(\frac{x^4}{24}+\frac{tx^2}{2}+\frac{t^2}{2}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \\
&\quad -\frac{x}{6}\sqrt{\frac{t}{\pi}}\left(\frac{x^2}{2}+5t\right)e^{-x^2/4t}\Bigg]\Bigg\} \\
&\quad + H(t-x\sqrt{\lambda})\Bigg\{\frac{G}{\lambda^2L^2}\Bigg[x\sqrt{\lambda}\int_{x\sqrt{\lambda}}^{t}\Bigg(\frac{e^{m_1(t-\zeta)}}{m_1^3(m_1-m_2)} \\
&\quad +\frac{e^{m_2(t-\zeta)}}{m_2^3(m_2-m_1)}+\frac{m_1^2+m_1m_2+m_2^2}{m_1^3m_2^3} \\
&\quad +\frac{(m_1+m_2)(t-\zeta)}{m_1^2m_2^2}+\frac{(t-\zeta)^2}{2m_1m_2}\Bigg)B(x,\zeta)\,d\zeta \\
&\quad + e^{-x/\sqrt{4\lambda}}\Bigg(\frac{e^{m_1(t-x\sqrt{\lambda})}}{m_1^3(m_1-m_2)}+\frac{e^{m_2(t-x\sqrt{\lambda})}}{m_2^3(m_2-m_1)} \\
&\quad +\frac{m_1^2+m_1m_2+m_2^2}{m_1^3m_2^3}+\frac{(m_1+m_2)(t-x\sqrt{\lambda})}{m_1^2m_2^2}+\frac{(t-x\sqrt{\lambda})^2}{2m_1m_2}\Bigg)\Bigg]\Bigg\}. \qquad \text{(A.3)}
\end{aligned}
$$

Appendix B

Letting $\lambda \to 0$ in (6.1) and then inverting for $f(t) = g(t) = H(t)$ and $P \neq 1$, we have

$$
\begin{aligned}
u(x,t) &= H(t)\Bigg\{ e^{-x/L}\Bigg[1+(e^{t/L^2}-1)(1-M_1L^2)+GL^2-\frac{GL^2}{1-P}(e^{t/L^2}-P\,e^{t/L^2P})\Bigg] \\
&\quad +\frac{(1-P)(1-M_1L^2)-GL^2}{1-P}\Bigg[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \\
&\quad -\frac{e^{t/L^2}}{2}\Bigg(e^{x/L}\operatorname{erfc}[v_+(x/L,t/L^2)] \\
&\quad + e^{-x/L}\operatorname{erfc}[v_-(x/L,t/L^2)]\Bigg)\Bigg]-\frac{G}{1-P}\Bigg[\left(\frac{x^2}{2}+t\right)
\end{aligned}
$$

$$\times \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - x\sqrt{\frac{t}{\pi}}\,\mathrm{e}^{-x^2/4t}\Bigg]$$

$$+ \frac{G}{1-P}\left[\left(P\frac{x^2}{2}+t\right)\mathrm{erfc}\left(\frac{x}{2}\sqrt{\frac{P}{t}}\right) - x\sqrt{\frac{Pt}{\pi}}\,\mathrm{e}^{-Px^2/4t}\right]$$

$$+ \frac{GPL^2}{1-P}\Bigg[\mathrm{erfc}\left(\frac{x}{2}\sqrt{\frac{P}{t}}\right) - \frac{\mathrm{e}^{t/(PL^2)}}{2}(\mathrm{e}^{x/L}\,\mathrm{erfc}[v_+(x/L, t/(PL^2))]$$

$$+\mathrm{e}^{-x/L}\,\mathrm{erfc}[v_-(x/L, t/(PL^2)])\Bigg]\Bigg\}. \qquad \text{(B.1)}$$

When $l_1 = l_2$ and $P = 1$ simultaneously, we cannot find u based on Fourier's heat law by letting $\lambda \to 0$ in Equation (6.1) and then inverting. We must find it directly by setting $l_1 = l_2 \equiv L$ in (2.17), $\lambda = 0$ in (2.18), and dropping the initial condition on θ_t in Equation (2.20). Thus, for $f(t) = g(t) = H(t)$, $l_1 = l_2 \equiv L$, and $P = 1$ u based on Fourier's heat law is given by

$$u(x,t) = H(t)\Bigg\{\mathrm{e}^{-x/L}\left[1 + (\mathrm{e}^{t/L^2}-1)(1-M_1L^2) + GL^2 + G\,\mathrm{e}^{t/L^2}(t-L^2)\right]$$

$$+(1-L^2(M_1+G))\Bigg[\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \frac{\mathrm{e}^{t/L^2}}{2}(\mathrm{e}^{x/L}\,\mathrm{erfc}[v_+(x/L, t/L^2)]$$

$$+\mathrm{e}^{-x/L}\,\mathrm{erfc}[v_-(x/L, t/L^2)])\Bigg] - \frac{G\,\mathrm{e}^{t/L^2}}{4}\Bigg[\mathrm{e}^{x/L}(2t+Lx)\mathrm{erfc}[v_+(x/L, t/L^2)]$$

$$+\mathrm{e}^{-x/L}(2t-Lx)\mathrm{erfc}[v_-(x/L, t/L^2)]\Bigg]$$

$$+\frac{GL\,\mathrm{e}^{t/L^2}}{4}\left[x\,\mathrm{e}^{x/L}\,\mathrm{erfc}[v_+(x/L, t/L^2)] - x\,\mathrm{e}^{-x/L}\,\mathrm{erfc}[v_-(x/L, t/L^2)]\right]$$

$$+\frac{G}{2}\Bigg[x\sqrt{\frac{t}{\pi}}\,\mathrm{e}^{-x^2/4t} - \frac{x^3\,\mathrm{e}^{-x^2/4t}}{2\sqrt{\pi t}} - x^2$$

$$\times\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) + \left(\frac{x^3}{2} + xt\right)\frac{\mathrm{e}^{-x^2/4t}}{\sqrt{\pi t}}\Bigg]\Bigg\}. \qquad \text{(B.2)}$$

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